MATH-14 (403(C)) TOPOLOGICAL DYNAMICS (1)

STUDY MATERIAL for the week ~~20th~~

(17th March to 20th March, 2020)

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STABLE AND UNSTABLE SETS

Let (X,d) be a compact metric space and $f: X \to X$ be a homeomorphism. For $x \in X$ and $\varepsilon > 0$, local stable set $W^s_\varepsilon(x,d)$ and local unstable set $W^u_\varepsilon(x,d)$ are defined by

$$W^s_\varepsilon(x,d) = \{ y \in X \mid d(f^n(x), f^n(y)) \leq \varepsilon, \ n \geq 0 \}$$

$$W^u_\varepsilon(x,d) = \{ y \in X \mid d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon, \ n \geq 0 \}.$$

Theorem: Let (X,d) be a compact metric space and $f: X \to X$ be a homeomorphism. Then f is expansive if and only if there exists $e > 0$ such that for all $\gamma > 0$ there exists an $N > 0$ such that for all $x \in X$ and $n \geq N$

$$\left.\begin{aligned} f^n(W^s_e(x,d)) &\subseteq W^s_\gamma(f^n(x), d) \\ f^{-n}(W^u_e(x,d)) &\subseteq W^u_\gamma(f^{-n}(x), d). \end{aligned}\right\} (*)$$

Proof. Suppose f is expansive with expansive constant e'. Assume that $(*)$ is not true. Then $\exists \ \gamma > 0$, $x_n, y_n \in X$, $m_n > 0$ such that $\lim_{n\to\infty} m_n = \infty$, $y_n \in W^s_e(x_n,d)$ and

$$d(f^{m_n}(x_n), f^{m_n}(y_n)) > \gamma. \longrightarrow (I)$$

Since $y_n \in W^s_e(x,d)$, we have

$$d(f^j(y_n), f^j(x_n)) \leq e, \ \forall j \geq 0.$$

Therefore

$$d(f^{i+m_n}(y_n), f^{i+m_n}(x_n)) \le e, \quad \forall (i+m_n) \geq 0$$
(taking $j = i+m_n$)

or $d(f^i(f^{m_n}(y_n)), f^i(f^{m_n}(x_n)) \le e, \quad \forall i \geq (-m_n) \rightarrow$ (II)

Since X is a compact metric space therefore sequences $\{f^{m_n}(y_n)\}$ and $\{f^{m_n}(x_n)\}$ have convergent subsequences. Without loss of generality, we can assume that $\{f^{m_n}(y_n)\} \to y$ and $\{f^{m_n}(x_n)\} \to x$ for some $x, y \in X$.

Since $\lim_{n\to\infty} m_n = \infty$ therefore from (II), we get

$$\cancel{d(f^i(y))} \; d(f^i(y), f^i(x)) \le e, \quad \forall i \in \mathbb{Z}$$
(using continuity of 'd')

Now, we show that $x \neq y$.

Note that $d(x,y) = d(\lim_{n\to\infty} f^{m_n}(x_n), \lim_{n\to\infty} f^{m_n}(y_n))$

$= \lim_{n\to\infty} d(f^{m_n}(x_n), f^{m_n}(y_n))$
($\because$ d is continuous)

$\geq \gamma$ (using (I))

> 0

therefore $x \neq y$.

Since $x \neq y$ and $d(f^n(x), f^n(y)) \le e, \forall n \in \mathbb{Z}$, we get a contradiction to the fact that f is expansive with expansive constant 'e'.

So (*) holds.

Conversely, suppose $\exists$ $e>0$ such that for all $\gamma>0$, $\exists N \in \mathbb{N}$ such that for all $x \in X$ and all $n \geq N$

$$\left.\begin{aligned} f^n(W_e^s(x,d)) &\subseteq W_\gamma^s(f^n(x),d) \\ f^{-n}(W_e^u(x,d)) &\subseteq W_\gamma^u(f^{-n}(x),d). \end{aligned}\right\} \quad (*)$$

We show that f is expansive with expansive constant 'e'.

Suppose $x, y \in X$ with $d(f^n(x), f^n(y)) \leq e$, $\forall n \in \mathbb{Z}$.
Using $(*)$, we get that for any $\gamma>0$,
$y \in W_\gamma^s(x,d)$ so $d(f^j(x), f^j(y)) \leq \gamma$, $\forall j \geq 0$

Taking $j=0$, we get $d(x,y) \leq \gamma$.
Since for any $\gamma>0$, $d(x,y) \leq \gamma$, we get that $d(x,y)=0$ i.e $x=y$.

Hence f is expansive with expansive constant 'e'.

Given $x \in X$, $y \in X$ is said to be forward asymptotic to x if $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ and $y \in X$ is said to be backward asymptotic to x if $d(f^n(x), f^n(y)) \to 0$ as $n \to -\infty$.

The stable and unstable sets of homeomorphism f are defined by

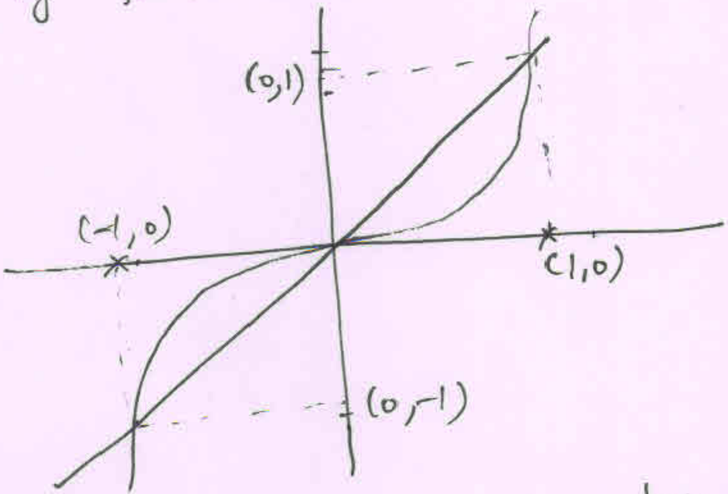
$$W^s(x,d) \equiv W^s(x) = \{y \in X \mid \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$$

i.e $W^s(x)$ is the set of all points forward asymptotic to x, and $W^u(x,d) \equiv W^u(x) = \{y \in X \mid \lim_{n\to -\infty} d(f^n(x), f^n(y)) = 0\}$

i.e $W^u(x)$ is the set of all points ~~forward~~ backward asymptotic to x.

Example

Consider the homeomorphism $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.



fixed points of f are given by $f(x) = x$ i.e $x^3 = x$ so $x = 0, \pm 1$. Note that the fixed points of f correspond to the points where the graph of f $\{(x, f(x)) \mid x \in X\}$ intersects the diagonal $\{(x,x) \mid x \in X\}$. The graph of f is monotone. On $(0,1)$, the graph of f lies below the diagonal and $f(x) < x$. Thus for $x \in (0,1)$, $x > f(x) > f^2(x) > \cdots > f^n(x) \cdots > 0$. Since this sequence is monotone, it must converge to a fixed point and so to '0'. (Note that a bounded monotone sequence of points on an orbit must converge to a fixed point)

Thus $(0,1) \subseteq W^s(0)$.

For backward iterates, $x < f^{-1}(x) < 1$ for $x \in (0,1)$. As $j \to -\infty$, $f^j(x)$ is monotonically increasing to 1, so $(0,1) \subseteq W^u(1)$.

By similar arguments $(-1,0) \subseteq W^s(0)$ and $(-1,0) \subseteq W^u(-1)$ since for $x \in (-1,0)$,

$$x < f(x) < f^2(x) < \cdots < f^n(x) \cdots < 0$$

and $$-1 < f^{-n}(x) < \cdots < f^{-2}(x) < f^{-1}(x) < x.$$

Since $W^s(0)$ contains a neighbourhood of '0', the fixed point '0' is an attracting fixed point.

For $x > 1$, $f^j(x)$ is monotonically increasing. If this forward orbit were bounded it would have to converge to a fixed point. Since there is no fixed point greater than 1, the orbit must go to infinity as j goes to infinity. As j goes to $-\infty$, $f^j(x)$ is monotonically decreasing to 1 so $W^u(1) \supseteq (1, \infty)$

Again for $x < -1$, $f^j(x)$ is monotonically decreasing to $-\infty$ as j goes to infinity and $f^j(x)$ is monotonically increasing as $j \to -\infty$ so $W^u(-1) \supseteq (-\infty, -1)$. The following list summarizes the stable and unstable sets:

$$W^s(0) = (-1, 1), \quad W^u(0) = \{0\}$$

$$W^s(\pm 1) = \{\pm 1\}, \quad W^u(1) = (0, \infty)$$

$$W^u(-1) = (-\infty, 0)$$

Since $W^u(1)$ is a neighbourhood of 1 (and $W^s(1)$ is not), the iterates of points near 1 move away and the fixed point 1 is repelling. Similarly -1 is repelling.

[For a continuous function, the forward orbit of a point a is the set $O^+(a) = \{f^k(a) \mid k \geq 0\}$. If f is invertible then backward orbit is defined as $O^-(a) = \{f^k(a) \mid k \leq 0\}$.

Theorem! Let (X,d) be a compact metric space and $f: X \to X$ be an expansive homeomorphism. Then for $\varepsilon > 0$, a number less than an expansive constant for f, we have

$$W^s(x,d) = \bigcup_{n \geq 0} f^{-n} W^s_\varepsilon(f^n(x), d)$$

$$W^u(x,d) = \bigcup_{n \geq 0} f^{n} W^u_\varepsilon(f^{-n}(x), d).$$

Proof! If $y \in \bigcup_{n \geq 0} f^{-n} W^s_\varepsilon(f^n(x), d)$ then there is an $n \geq 0$ with $f^n(y) \in W^s_\varepsilon(f^n(x), d)$. Therefore, for $\gamma > 0$ there is an $N \geq 0$ such that for all $m \geq N$

$$f^{m+n}(y) \in f^m(W^s_\varepsilon(f^n(x), d)$$
$$\subseteq W^s_\gamma(f^{m+n}(x), d)$$

and hence
$$d(f^{m+n}(y), f^{m+n}(x)] \leq \gamma$$
for $m \geq N$.

Thus $\lim_{n \to \infty} d(f^n(y), f^n(x)) = 0$ i.e $y \in W^s(x,d)$.

Conversely, if $z \in W^s(x,d)$ and $\varepsilon > 0$, then we can find an $N \geq 0$ such that $d(f^n(x), f^n(y)) \leq \varepsilon$ for all $n \geq N$.

Therefore

$$d(f^j(f^N(x)), f^j(f^N(y)) \leq \varepsilon \text{, for all } j \geq 0.$$

Hence $f^N(y) \in W^s_\varepsilon(f^N(x), d)$ and so

$$y \in f^{-N} W^s_\varepsilon(f^N(x), d) \subseteq \bigcup_{n \geq 0} f^{-n} W^s_\varepsilon(f^n(x), d).$$

By similar arguments, one can prove the second relation

<u>Theorem:</u> If (X, d) is a compact metric space and $f: X \to X$ is continuous onto map, for some $x \in X$ and $m \geq 0$,
$f^m(W^s(x,d)) \cap W^s(x,d) \neq \phi$
then X contains a periodic point.

<u>Proof:</u> Take $y \in W^s(x,d) \cap f^m(W^s(x,d))$
and put $z = f^{-m}(y)$. Then
$f^m(z) \in W^s(x,d) = W^s(z,d)$.
Therefore $\lim_{n \to \infty} d(f^n f^m(z), f^n(z)) = 0$.
Since $\{f^n(z)\}$ is a sequence of points in compact metric space X, there exists a subsequence $\{f^{m_n}(z)\} \to w$, say as $n \to \infty$.
Therefore
$$d(w, f^m(w)) = \lim_{n \to \infty} d(f^m f^{m_n}(z), f^{m_n}(z))$$
$$= \lim_{n \to \infty} d(f^{m_n} f^m(z), f^{m_n}(z)) = 0.$$

Hence $f^m(w) = w$.

Let $f: X \to X$ be a homeomorphism of a Compact metric Space X. Then f is said to have a Canonical Coordinate if for each ~~for~~ $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d(x,y) \leq \delta$ implies $W^s_\varepsilon(x,d) \cap W^u_\varepsilon(y,d) \neq \phi$.

<u>Theorem</u>: If $f: X \to X$ has POTP then f has canonical coordinates.

<u>Proof</u>: For $\varepsilon > 0$, let $\delta > 0$ be a number with the property of the POTP. We show that for this δ, $d(x,y) < \delta$ implies $W^s_\varepsilon(x,d) \cap W^u_\varepsilon(y,d) \neq \phi$.

Suppose $x, y \in X$ such that $d(x,y) < \delta$. Define sequence $\{x_i\}_{i \in \mathbb{Z}}$ of X as follows: $x_i = f^i(x)$ for $i \geq 0$ and $x_i = f^i(y)$ for $i < 0$. Then $\{x_i\}_{i \in \mathbb{Z}}$ is a δ-pseudo orbit for f. Hence by POTP of f, $\exists\, z \in X$ satisfying $d(f^i(z), x_i) < \varepsilon$ for each $i \in \mathbb{Z}$.

Note that $z \in W^s_\varepsilon(x,d) \cap W^u_\varepsilon(y,d)$.

$d(f^i(z), x_i) < \varepsilon$, $\forall i \geq 0$ implies and $x_i = f^i(x)$ for each $i \geq 0$ which implies $d(f^i(z), f^i(x)) < \varepsilon$, $\forall i \geq 0$ $d(f^i(z), x_i) < \varepsilon$, for $z \in W^s_\varepsilon(x,d)$. Also, $d(f^i(z), x_i) < \varepsilon$, for each $i < 0$ and $x_i = f^i(y)$, for $i < 0$ implies $d(f^i(z), f^i(y)) < \varepsilon$, $i < 0$ so $z \in W^u_\varepsilon(y,d)$.